J. Math. Pures Appl. 79, 1 (2000) 1–20

INVARIANTS FOR A CLASS OF EQUIVARIANT IMMERSIONS OF THE UNIVERSAL COVER OF A COMPACT RIEMANN SURFACE INTO A PROJECTIVE SPACE

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Manuscript received 30 April 1999

ABSTRACT. – We consider the space of all holomorphic immersions of the universal cover of a compact hyperbolic Riemann surface into $\mathbb{CP}^n$ satisfying the condition that at every point of the image, the order of contact of the image with any hyperplane in $\mathbb{CP}^n$ is at most $n-1$. A further restriction that is imposed states as follows: there exists a homomorphism of the Galois group for the universal cover into $\mathrm{GL}(n+1,\mathbb{C})$ such that the map from the universal cover to $\mathbb{CP}^n$ is equivariant for the actions of the Galois group. To each such immersion we associate a i-form on the compact Riemann surface for each $i \in [3, n+1]$, and also associate a projective structure on the Riemann surface. The resulting map from the space of all immersions surjects onto the target space. Moreover, this map gives a bijective correspondence between the target space and the space of all equivalence classes of immersions, where the equivalence relation identifies an immersion with any other immersion obtained by composing it with an automorphism of $\mathbb{CP}^n$. © 2000 Éditions scientifiques et médicales Elsevier SAS

Keywords: Projective structure, Differential operator, Projective embedding

1. Introduction

Let X be a compact connected hyperbolic Riemann surface, and let $\pi : \widetilde{X} \to X$ be a universal cover of X with Galois group Γ. We will call a map γ from $\widetilde{X}$ into $\mathbb{CP}^n$ to be everywhere locally nondegenerate if for any $y \in \widetilde{X}$ and any hyperplane H passing through $\gamma(y)$, the order of contact of $\widetilde{X}$ with H is at most $n-1$.

The objects of study here are pairs of the form (ρ, γ), where ρ is a homomorphism from Γ to $\mathrm{GL}(n+1,\mathbb{C})$ and γ is a Γ-equivariant everywhere locally nondegenerate map from $\widetilde{X}$ to $\mathbb{CP}^n$; the action of Γ on $\mathbb{CP}^n$ is defined by composing ρ with the natural action of $\mathrm{GL}(n+1,\mathbb{C})$ on $\mathbb{CP}^n$.

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The main result established here is the existence of a canonical surjective map from the space of all pairs of the above type to the Cartesian product:

$$\mathfrak{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0\big(X,\, K_X^{\otimes i}\big)\right),$$

where $\mathfrak{P}(X)$ denotes the space of all projective structures on X (Theorem 5.5). We recall that $\mathfrak{P}(X)$ is an affine space for the vector space $H^0(X,\, K_X^{\otimes 2})$. In other words, to each pair (ρ, γ) of the above type, Theorem 5.5 associates invariants consisting of projective structure on X and a holomorphic j-form on X for each $j \in [3, n+1]$. When $n = 1$, the above mentioned map reduces to the tautological identification of a projective structure with the corresponding developing map of $\widetilde{X}$ to $\mathbb{CP}^1$. Given a projective structure P on X, the developing map, which is unique up to an automorphism of $\mathbb{CP}^1$, is determined by the condition that the coordinate function on $\widetilde{X}$ defined by it is compatible with the projective structure on $\widetilde{X}$ induced by P.

Also, to each element $\omega \in \mathfrak{P}(X) \times (\bigoplus_{i=3}^{n+1} H^0(X,\, K_X^{\otimes i}))$ there is a canonically associated pair (ρ, γ) such that the set of invariants associated to (ρ, γ) is the element ω we started with.

Two pairs (ρ, γ) and (ρ', γ'), of the above type, will be called equivalent if there is a linear isomorphism F of $\mathbb{CP}^n$ such that $F \circ \gamma = \gamma'$ and the homomorphism from Γ to $\mathrm{GL}(n+1, \mathbb{C})$ given by $F\rho F^{-1}$ coincides with the one given by ρ'. The equivalence class represented by a pair (ρ, γ) is determined by the map γ itself. In other words, any other pair (ρ', γ) is equivalent to (ρ, γ); this is a consequence of the non-degeneracy condition satisfied by γ.

The above mentioned map from the space of all pairs (ρ, γ) to

$$\mathfrak{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0\big(X,\, K_X^{\otimes i}\big)\right),$$

factors through the space of all equivalence classes of pairs. Furthermore, the resulting map from the space of all equivalence classes of pairs is bijective (Theorem 5.5).

The main inputs of this paper are provided by an earlier work [2]. One such input is a decomposition of the space of a certain class of differential operators on X established in [2] (which is reproduced here in (3.6)) that is used in the construction of the invariants carried out in Theorem 5.5.

The space $\mathfrak{P}(X) \times (\bigoplus_{i=3}^{n+1} H^0(X,\, K_X^{\otimes i}))$ arises in several contexts. Let $p_i,\ i = 1, 2$, denote the projection of $X \times X$ onto the i-th factor. The diagonal divisor in $X \times X$ with be denoted by Δ. The restriction of the line bundle

$$L := p_1^* K_X \otimes p_2^* K_X \otimes \mathcal{O}_{X \times X}(2\Delta)$$

to the nonreduced divisor 2Δ has a canonical trivialization. In [3] it was shown that the space of all trivializations of L over $(n+2)\Delta$ restricting to the canonical trivialization over 2Δ, is canonically identified with $\mathfrak{P}(X) \times (\bigoplus_{i=3}^{n+1} H^0(X,\, K_X^{\otimes i}))$.

Let ξ be a line bundle over X with $\xi^{\otimes 2} = T_X$. There is a natural identification of 3Δ with the $3\Delta_{\mathbb{P}}$, where $\Delta_{\mathbb{P}}$ denotes the divisor on the projectivized jet bundle $\mathbb{P}(J^1(\xi))$ defined by the obvious homomorphism $J^1(\xi) \to \xi$ [4]. Pierre Deligne showed that $\mathcal{P}(X)$ is naturally identified with the space of extensions of the above mentioned identification to an identification between

4Δ and $4\Delta_{\mathbb{P}}$ [4]. More generally, the space

$$\mathfrak{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0\big(X,\ K_X^{\otimes i}\big) \right)$$

is identified with the space of all isomorphisms of $(n+3)\Delta$ with $(n+3)\Delta_{\mathbb{P}}$ extending the natural identification of 3Δ with $3\Delta_{\mathbb{P}}$ [3].

2. Flat connections with a special second fundamental form

Let X be a compact connected Riemann surface of genus g, with $g \geqslant 2$. Let E be a holomorphic vector bundle of rank r over X, equipped with a holomorphic connection D. In other words,

$$D : E \to K_X \otimes E \tag{2.1}$$

is a $\mathbb{C}$-linear homomorphism of sheaves satisfying the Leibniz identity $D(f \cdot s) = \mathrm{d}f \otimes s + f \cdot D(s)$, where f is a local holomorphic function and s is a local holomorphic section of E. Consequently, the connection $D + \overline{\partial}_E$ on E, where $\overline{\partial}_E$ is the Dolbeault operator defining the holomorphic structure of E, is flat.

Take a line subbundle, say L, of E. Define F_2 to be the unique coherent subsheaf of E such that $D(L) = K_X \otimes F_2$, where D as in (2.1). The Leibniz identity ensures that L is a subsheaf of F_2. Indeed, for a local section s of L, defined around a point $x \in X$, and a local holomorphic function f with $f(x) = 0$, we have $D(f \cdot s)(x) = \mathrm{d}f(x) \otimes s(x)$.

Set $F_0 = 0$ and $F_1 = L$, and define the coherent subsheaf F_i, $i \geqslant 2$, of E inductively to be the unique coherent subsheaf of E such that $D(F_{i-1}) = K_X \otimes F_i$. As before, using the Leibniz identity we deduce that F_{i-1} is a subsheaf of F_i.

The rank of F_i will be shown now to be at most i. Consider the homomorphism:

$$\overline{D} : L \to K_X \otimes (F_2/L), \tag{2.2}$$

it sends any local section s of L to the projection of $D(s)$ in $K_X \otimes (F_2/L)$. This operator $\overline{D}$ is known as the *second fundamental form* of L.

The Leibniz condition implies that $\overline{D}$ is $\mathcal{O}_X$-linear. Therefore, the rank of the coherent sheaf $K_X \otimes (F_2/L)$ is at most one. This means that the rank of F_2 is at most two. Now it follows inductively that the rank of F_{i+1}/F_i is at most one for every $i \geqslant 2$. Indeed, consider the surjective $\mathcal{O}_X$-linear homomorphism

$$\overline{D} : F_i/F_{i-1} \to K_X \otimes (F_{i+1}/F_i) \tag{2.3}$$

induced by D exactly as in (2.2). This yields that the rank of F_{i+1}/F_i is at most the rank of F_i/F_{i-1}.

Now an assumption on the connection D will be made:

ASSUMPTION 2.4. – *There is a line subbundle L of the flat vector bundle E such that each F_i defined above, where $i \in [1, r]$, is a subbundle of E of rank i. In particular, we have $F_r = E$.*

Actually, the condition $F_r = E$ is equivalent to the above assumption. Indeed, it is easy to show that if $F_r = E$, then each F_i must be a subbundle of E of rank i.

The above assumption implies that the homomorphism $\overline{D}$ in (2.3) is an isomorphism, i.e.,

$$F_i/F_{i-1} = K_X \otimes (F_{i+1}/F_i).$$

Since the degree $\deg(E) = 0$, the above equality implies that $\deg(F_i/F_{i-1}) = (g-1)(r-2i+1)$.

As the degree $\deg(K_X) > 0$, the inequality $\deg(F_i/F_{i-1}) > \deg(F_{i+1}/F_i)$ is valid. From this inequality it follows immediately that the filtration

$$F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = E \tag{2.5}$$

is the Harder–Narasimhan filtration of E. Since the Harder–Narasimhan filtration is unique, the choice of the line bundle L in Assumption 2.4 is unique. The details of the Harder–Narasimhan filtration of a vector bundle can be found in [7].

The following proposition describes the extension F_i of F_i/F_{i-1} by F_{i-1}:

PROPOSITION 2.6. – *Let E be a flat vector bundle satisfying Assumption* 2.4. *For any $i \geqslant 2$, the homomorphism*

$$p : H^1\big(X, \mathrm{Hom}(F_i/F_{i-1}, F_{i-1})\big) \to H^1\big(X, \mathrm{Hom}(F_i/F_{i-1}, F_{i-1}/F_{i-2})\big),$$

induced by the natural projection of F_{i-1} on F_{i-1}/F_{i-2}, is injective. Furthermore, the following is valid:

$$H^1\big(X, \mathrm{Hom}(F_i/F_{i-1}, F_{i-1}/F_{i-2})\big) = \mathbb{C}.$$

The extension

$$0 \to F_{i-1} \to F_i \to F_i/F_{i-1} \to 0$$

is the unique nontrivial extension.

Proof. – Since $H^1(X, \mathrm{Hom}(F_i/F_{i-1}, F_{i-j}/F_{i-j-1})) = H^1(X, K_X^{\otimes j}) = 0$ if $j \geqslant 2$, repeatedly using the exact sequence

$$H^1\left(X, \mathrm{Hom}\left(\frac{F_i}{F_{i-1}}, F_{i-j-1}\right)\right) \longrightarrow H^1\left(X, \mathrm{Hom}\left(\frac{F_i}{F_{i-1}}, F_{i-j}\right)\right) \longrightarrow H^1\left(X, \mathrm{Hom}\left(\frac{F_i}{F_{i-1}}, \frac{F_{i-j}}{F_{i-j-1}}\right)\right),$$

we conclude that $H^1(X, \mathrm{Hom}(F_i/F_{i-1}, F_{i-2})) = 0$. Therefore, the homomorphism p in the statement of the proposition must be injective.

Since

$$H^1\big(X, \mathrm{Hom}(F_i/F_{i-1}, F_{i-1}/F_{i-2})\big) = H^1(X, K_X) = \mathbb{C},$$

to complete the proof it is enough to show that the exact sequence

$$0 \to F_{i-1} \to F_i \to F_i/F_{i-1} \to 0$$

of vector bundles does not split if $i \geqslant 2$.

Assume that the above exact sequence actually splits. Then the earlier observation that $H^1(X, \mathrm{Hom}(F_k/F_{k-1}, F_{k-2})) = 0$ for every $k \geqslant 2$ implies that the exact sequence

$$0 \to F_{i-1} \to E \to E/F_{i-1} \to 0$$

of vector bundles splits. In such a situation, the holomorphic connection on E induces a holomorphic connection on F_{i-1}. This would imply that $\deg(F_{i-1}) = 0$, which contradicts the fact that $\deg(F_{i-1}) > 0$ for all $i \in [1, r-1]$. This completes the proof of the proposition. □

Let $\mathcal{L}$ denote the line bundle E/F_{r-1}. The vector bundle E will be shown to be a jet bundle of $\mathcal{L}$. We quickly recall the definition of a jet bundle.

For a holomorphic vector bundle V on X and a positive integer n, the n-th order *jet bundle* of V, denoted by $J^n(V)$, is defined to be the vector bundle over X corresponding to the following direct image:

$$J^n(V) := p_{1*}\left(\frac{p_2^* V}{p_2^* V \otimes \mathcal{O}_{X\times X}(-(n+1)\Delta)}\right)$$

on X, where Δ is the diagonal divisor in $X \times X$ and p_i, $i = 1, 2$, is the projection of $X \times X$ onto the i-th factor. The natural inclusion of the sheaf $p_2^* V \otimes \mathcal{O}_{X\times X}(-(n+1)\Delta)$ in $p_2^* V \otimes \mathcal{O}_{X\times X}(-n\Delta)$ induces an exact sequence:

$$0 \to K_X^{\otimes(n+1)} \otimes V \to J^{n+1}(V) \to J^n(V) \to 0. \tag{2.7}$$

The inclusion $K_X^{\otimes(n+1)} \otimes V \to J^{n+1}(V)$ is constructed by using the inclusion homomorphism

$$K_X^{\otimes(n+1)} \to J^{n+1}(\mathcal{O}_X),$$

which at any $x \in X$ is defined by $(\mathrm{d}f)^{\otimes(n+1)} \mapsto f^{n+1}$, where f is any holomorphic function with $f(x) = 0$.

For any integer $i \geqslant 0$, the flat connection D on E gives a natural homomorphism

$$\Phi(i) : E \to J^i(\mathcal{L}), \tag{2.8}$$

which will be described now. Let ψ denote the natural projection of E onto its quotient $\mathcal{L}$. For any $x \in X$ and $v \in E_x$, let $\overline{v}$ denote the unique local flat section of E defined over a simply connected analytic neighborhood of x such that $\overline{v}(x) = v$. Now, $\psi(\overline{v})$ is a local section of $\mathcal{L}$. The homomorphism $\Phi(i)$ sends the vector v to the element of the fiber $J^i(\mathcal{L})_x$ over x representing the local section $\psi(\overline{v})$ of $\mathcal{L}$.

THEOREM 2.9. – *Let E be a flat vector bundle satisfying Assumption* 2.4. *The vector bundle homomorphism $\Phi(r-1)$ from E to $J^{r-1}(\mathcal{L})$ constructed in* (2.8) *is an isomorphism.*

Proof. – Using induction on i, where $i \in [0, r-1]$, it will be proved that $\Phi(i)$ is surjective. Since $\mathrm{rank}(J^{r-1}(\mathcal{L})) = r = \mathrm{rank}(E)$, this would prove the theorem.

Since $\Phi(0)$ coincides with ψ, it is surjective.

To prove that $\Phi(1)$ is surjective, take any $x \in X$ and any $w \in J^1(\mathcal{L})_x$. Let w_0 denote the image of w by the natural projection of $J^1(\mathcal{L})_x$ onto $\mathcal{L}_x$ defined in (2.7) with $n = 0$. Take any $v \in E_x$ such that $\psi(v) = w_0$. Now the element

$$\widehat{w} := w - \Phi(1)(v) \in J^1(\mathcal{L})_x \tag{2.10}$$

is contained in the image of $(K_X \otimes \mathcal{L})_x$ in $J^1(\mathcal{L})_x$ by the natural inclusion defined in (2.7). The homomorphism $\overline{D}$, defined in (2.3), is an isomorphism of F_{r-1}/F_{r-2} with $K_X \otimes \mathcal{L}$. Take a vector $v' \in (F_{r-1})_x$ such that the image of v' in $(F_{r-1}/F_{r-2})_x$ corresponds to the element $\widehat{w}$, defined in (2.10), by the isomorphism between $(F_{r-1}/F_{r-2})_x$ and $(K_X \otimes \mathcal{L})_x$. Let $\widetilde{v'}$ be a local section of F_{r-1} in a neighborhood of x such that $\widetilde{v'}(x) = v'$. Let $\overline{v'}$ denote the local flat section of E constructed by taking parallel translations of v'. We note that $\psi(\overline{v'} - \widetilde{v'})$ and $\psi(\overline{v'})$ are represented by the same element of $J^1(\mathcal{L})_x$. Also, since the section $\overline{v'}$ is flat, we conclude that the image of the local section $D(\widetilde{v'})$, of $K_X \otimes E$, in $(K_X \otimes \mathcal{L})_x$ by the composition

$$K_X \otimes E \xrightarrow{\mathrm{Id}_{K_X} \otimes \psi} K_X \otimes \mathcal{L} \longrightarrow (K_X \otimes \mathcal{L})_x,$$

of homomorphisms of sheaves, where the second map is the restriction to x, coincides with the element in $(K_X \otimes \mathcal{L})_x$ corresponding to $D(\widetilde{v'} - \overline{v'})$ defined in the same way. Therefore, the image of v' by the following composition of homomorphisms

$$F_{r-1} \longrightarrow F_{r-1}/F_{r-2} \xrightarrow{\overline{D}} K_X \otimes \mathcal{L},$$

where the first homomorphism is the natural projection, coincides with $\Phi(1)(v')$. For a local section s of E, defined in a neighborhood of x, with $s(x) = 0$, the element of $J^1(E)_x$ representing s is an element of the subspace $(K_X \otimes E)_x \subset J^1(E)_x$. From the Leibniz identity satisfied by the operator D it follows that the element of $(K_X \otimes E)_x$ representing s coincides with $D(s)(x)$. Therefore, setting $s = \widetilde{v'} - \overline{v'}$, from the definition of v' and (1.9) it is deduced that $\Phi(1)(v + v') = w$.

To prove the surjectivity of every $\Phi(i)$, $i \in [0, r-1]$, by induction, fix $i \in [2, r-1]$, and assume that $\Phi(j)$ is surjective for all $j \in [0, i-1]$.

Take any $x \in X$ and any $w \in J^i(\mathcal{L})_x$. Let w_0 denote the image of w in $J^{i-1}(\mathcal{L})_x$ by the natural projection of $J^i(\mathcal{L})_x$ onto $J^{i-1}(\mathcal{L})_x$. Take any $v \in E_x$ such that the element $\Phi(i-1)(v)$ of $J^{i-1}(\mathcal{L})_x$ coincides with w_0. The existence of such a vector v is ensured by the induction hypothesis. Now, we have:

$$\widehat{w} := w - \Phi(i)(v) \in K_X^{\otimes i} \otimes \mathcal{L} \subset J^i(\mathcal{L})_x. \tag{2.11}$$

Recall that the operator $\overline{D}$ in (2.3) gives an isomorphism between F_k/F_{k-1} and $K_X \otimes (F_{k+1}/F_k)$ for every $k \in [1, r-1]$. Composing this operator i times, we get an isomorphism:

$$\widetilde{D} := \overline{D}^i : F_{r-i}/F_{r-i-1} \to K_X^{\otimes i} \otimes (F_r/F_{r-1}) = K_X^{\otimes i} \otimes \mathcal{L}. \tag{2.12}$$

Take any $v' \in (F_{r-i})_x$ such that the image of v' in the quotient space $(F_{r-i}/F_{r-i-1})_x$ coincides with the element $\widehat{w} \in K_X^{\otimes i} \otimes \mathcal{L}$, obtained in (2.11), by the isomorphism $\widetilde{D}$ constructed in (2.12).

As before, let $\overline{v'}$ be the local flat section of E with $\overline{v'}(x) = v'$. From a repeated application of the Leibniz identity it follows that the vector $\widehat{w} \in J^i(\mathcal{L})_x$ represents the section $\psi(\overline{v'})$ of $\mathcal{L}$, where $\widehat{w}$ is defined in (2.11). Now, from (2.11) it follows immediately that

$$\Phi(i)\big(v + \overline{v'}\big) = w.$$

This completes the proof of the theorem. $\square$

If we identity E with $J^{r-1}(\mathcal{L})$ using $\Phi(r-1)$, then any $\Phi(j)$, where $j \in [0, r-1]$, coincides with the natural projection of $J^{r-1}(\mathcal{L})$ onto $J^j(\mathcal{L})$ constructed in (2.7). In view of Theorem 2.9, this is indeed evident from the construction of $\Phi(i)$.

Since $\deg(E) = \deg(J^{r-1}(\mathcal{L})) = 0$, and

$$\deg\big(J^{r-1}(\mathcal{L})\big) = r \cdot \deg(\mathcal{L}) + \sum_{i=1}^{r-1} \deg\big(K_X^{\otimes i}\big),$$

we conclude that

$$\deg(\mathcal{L}) = (r-1)(1-g).$$

If $\mathcal{L}'$ is another line bundle over X of degree $(r-1)(1-g)$, then the degree zero line bundle $\mathcal{L}^* \otimes \mathcal{L}'$ has a unique unitary flat connection. Any local section s' of $\mathcal{L}'$ can be expressed as $s \otimes t$, where s is a local section of $\mathcal{L}$ and t is a flat local section of $\mathcal{L}^* \otimes \mathcal{L}'$. Furthermore, if $s \otimes t = s_0 \otimes t_0$, where s is another local section of $\mathcal{L}$ and t is another flat local section of $\mathcal{L}^* \otimes \mathcal{L}'$, then evidently $s = \lambda s_0$, where λ is a constant scalar. Consequently, we have a natural identification of $J^i(\mathcal{L}) \otimes \mathcal{L}^* \otimes \mathcal{L}'$ with $J^i(\mathcal{L}')$ for any $i \geqslant 0$.

The above observation combines with Theorem 2.9 to give the following corollary.

COROLLARY 2.13. – *If E and E' are two flat vector bundles satisfying Assumption* 2.4, *then E' is canonically isomorphic to $E \otimes \zeta$, where ζ is a line bundle over X of degree zero. The line bundle ζ coincides with $\mathcal{L}' \otimes \mathcal{L}^*$, where $\mathcal{L}$ (respectively, $\mathcal{L}'$) is the smallest quotient of E (respectively, E') for the Harder–Narasimhan filtration of E (respectively, E'). In particular, the two projective bundles $\mathbb{P}(E)$ and $\mathbb{P}(E')$ are canonically isomorphic.*

It should be emphasized that the isomorphism between $\mathbb{P}(E)$ and $\mathbb{P}(E')$ in Corollary 2.13 depends on the flat connections. More precisely, if D and D' are two flat connections on the same vector bundle E satisfying Assumption 2.4, then the corresponding automorphism of $\mathbb{P}(E)$ need not coincide with the identity automorphism of $\mathbb{P}(E)$.

It was observed earlier that the filtration $\{F_i\}$ is the Harder–Narasimhan filtration of the vector bundle E. Another consequence of this observation is that the isomorphism between E' and $E \otimes \zeta$ in Corollary 2.13 preserves the flag.

Since the Harder–Narasimhan filtration of $J^n(\mathcal{L})$ coincides with the one obtained from the filtration

$$J^n(\mathcal{L}) \to J^{n-1}(\mathcal{L}) \to J^{n-2}(\mathcal{L}) \to \cdots \to J^1(\mathcal{L}) \to \mathcal{L}, \tag{2.14}$$

the uniqueness of the Harder–Narasimhan filtration produces the following consequence of Theorem 2.9.

COROLLARY 2.15. – *The isomorphism $\Phi(r-1)$ induces an isomorphism of the quotient vector bundle E/F_i of E with the quotient vector bundle $J^{r-i-1}(\mathcal{L})$ of $J^{r-1}(\mathcal{L})$.*

The following lemma says that either every holomorphic connection or no holomorphic connection on any given holomorphic vector bundle satisfies Assumption 2.4.

LEMMA 2.16. – *Let E be a rank r holomorphic vector bundle which admits a holomorphic connection that satisfies Assumption* 2.4. *If D is any holomorphic connection on E, then D satisfies Assumption* 2.4.

Proof. – Let

$$F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = E$$

be the Harder–Narasimhan filtration of E. Consider the vector bundle homomorphism

$$\overline{D}_{1,r} : F_1 \to K_X \otimes (E/F_{r-1}),$$

which sends a local section s of F_1 to the projection of $D(s)$ on $K_X \otimes (E/F_{r-1})$. Assuming $r \geqslant 3$, since

$$\deg(F_1) = (r-1)(g-1) > \deg\big(K_X \otimes (E/F_{r-1})\big) = (3-r)(g-1),$$

we conclude that $\overline{D}_{1,r} = 0$. Therefore, $D(F_1) \subset K_X \otimes F_{r-1}$.

We repeat the above argument for the vector bundle homomorphism:

$$\overline{D}_{1,r-1} : F_1 \to K_X \otimes (F_{r-1}/F_{r-2}),$$

and conclude that $\overline{D}_{1,r-1} = 0$ if $r \geqslant 4$. Consequently, if $r \geqslant 4$, then the inclusion $D(F_1) \subset K_X \otimes F_{r-2}$ is valid.

Repeating the above argument $r-2$ times we conclude that $D(F_1) \subset K_X \otimes F_2$. But $\deg(F_1) = (r-1)(g-1) \neq 0$. So, F_1 does not admit a flat connection, and hence the vector bundle homomorphism

$$\overline{D}_{1,2} : F_1 \to K_X \otimes (F_2/F_1)$$

must be nonzero. Now, since $\deg(F_1) = \deg(K_X \otimes (F_2/F_1))$, the homomorphism $\overline{D}_{1,2}$ must be an isomorphism.

Assuming $r \geqslant 3$, consider the vector bundle homomorphism

$$\overline{D}_{2,r} : F_2/F_1 \to K_X \otimes (E/F_{r-1}),$$

which sends the projection of a local section s of F_2 in F_2/F_1 to the projection of $D(s)$ in $K_X \otimes (E/F_{r-1})$. This map is well defined since $D(F_1) \subset K_X \otimes F_2$. From degree consideration as above, $\overline{D}_{2,r} = 0$ if $r \geqslant 4$. Therefore, $D(F_2) \subset K_X \otimes F_{r-1}$. Repeatedly using this argument we conclude by induction that $D(F_2) \subset K_X \otimes F_3$. Since $\deg(F_2) = (2g-2)(r-2) \neq 0$, the vector bundle F_2 does not admit a flat connection. Therefore, the homomorphism

$$\overline{D}_{2,3} : F_2/F_1 \to K_X \otimes (F_3/F_2),$$

which sends the projection of a local section s of F_2 in F_2/F_1 to the projection of $D(s)$ in $K_X \otimes (F_3/F_2)$, must be nonzero. Now, since $\deg(F_2/F_1) = \deg(K_X \otimes (F_3/F_2))$, the homomorphism $\overline{D}_{2,3}$ must be an isomorphism.

Using induction on i and following the above arguments it is straight-forward to deduce that $D(F_i) = K_X \otimes F_{i+1}$ for all $i \in [1, r-1]$ and that the second fundamental form gives an isomorphism between F_i/F_{i-1} and $K_X \otimes (F_{i+1}/F_i)$. This completes the proof of the lemma. □

The above lemma has the following consequence.

COROLLARY 2.17. – *Let E be a holomorphic vector bundle, admitting a holomorphic connection D that satisfies Assumption* 2.4. *Then E is indecomposable.*

Proof. – Assume that $E = E_1 \oplus E_2$. The flat connection D on E induces a flat connection on both E_1 and E_2. The induced flat connection on E_i will be denoted by D_i. Let D' denote the flat connection $D_1 \oplus D_2$ on E, defined using the direct sum decomposition. From Lemma 2.16 it follows that D' satisfies Assumption 2.4.

Since the line subbundle F_1 of E, as in (2.5), is the maximal semistable subbundle, we have:

$$H^0\big(X, \operatorname{Hom}(F_1, E/F_1)\big) = 0.$$

Therefore, either F_1 is a subbundle of E_1, or F_1 is a subbundle of E_2.

Since Assumption 2.4 is valid for D', there is no proper flat subbundle, with respect to the connection D', of E that contains the line subbundle F_1. Therefore, if $F_1 \subseteq E_i$, then E_i must coincide with E. This completes the proof. □

A connection ∇ on a vector bundle V is called *irreducible* if there is no nonzero proper subbundle of V which is left invariant by ∇. The following proposition says that a connection satisfying Assumption 2.4 is irreducible.

PROPOSITION 2.18. – *Let D be a holomorphic connection, satisfying Assumption* 2.4*, on a holomorphic vector bundle E over X. Then the connection D is irreducible.*

Proof. – Let W be a holomorphic subbundle of rank k, where $k \geqslant 1$, of the rank r vector bundle E, such that W is left invariant by the connection D. Consider the filtration $\{F_i\}_{i\in[1,r]}$ of E given in (2.5). From the rank considerations we conclude that the intersection $W \cap F_{r-k+1}$ has rank at least one.

From Assumption 2.4 it follows that for a subbundle $V \subseteq E$, left invariant by D, if the intersection $V \cap F_j$ is of rank at least one, then the quotient of the following composition of homomorphisms

$$V \to E \to E/F_{j-1}$$

must be a torsion sheaf, i.e., the image of V in E/F_{j-1} is of rank $r - j + 1$. Indeed, this is immediately obtained by using the given fact that the homomorphism $\overline{D} : F_i/F_{i-1} \to K_X \otimes (F_{i+1}/F_i)$, constructed in (2.3), is an isomorphism for $i \geqslant j$.

Consequently, we have:

$$\deg(W) = \deg(E/F_{r-k}) - \dim T \leqslant \deg(E/F_{r-k}),$$

where T is the torsion sheaf obtained by taking the quotient of E/F_{r-k} by W. If $r - k \geqslant 1$, then $\deg(E/F_{r-k}) < 0$. On the other hand, since W has a holomorphic connection induced by D, we have $\deg(W) = 0$.

Therefore, we have $r - k = 0$. In other words, we have $E = W$. This completes the proof of the proposition.

An alternative way of seeing the validity of the proposition is as follows:

If $0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{k-1} \subset W_k = W$ is the Harder–Narasimhan filtration of a vector bundle W equipped with a flat connection, then

$$\frac{\deg(W_i/W_{i-1})}{\operatorname{rank}(W_i/W_{i-1})} - \frac{\deg(W_{i+1}/W_i)}{\operatorname{rank}(W_{i+1}/W_i)} \leqslant 2g - 2, \tag{2.19}$$

for every $i \in [1, k-1]$. The above inequality (2.19) follows from the combination of the fact that any direct summand V of W has degree zero (since it has an induced flat connection) and the fact that if V' is a semistable vector bundle over X with $\deg(V')/\operatorname{rank}(V') > 2g - 2$, then $H^1(X, V') = 0$.

Using the inequality (2.19) it is easy to deduce that the connection D is irreducible. □

Let $\pi : \widetilde{X} \to X$ be a universal cover of X. The group of deck transformations of the covering map π will be denoted by Γ.

For a flat vector bundle bundle E over X satisfying Assumption 2.4, consider the pull back $\pi^*\mathbb{P}(E)$ to $\widetilde{X}$ of the flat projective bundle $\mathbb{P}(E)$ over X consisting of all lines in E. Since $\widetilde{X}$ is simply connected, the flat projective bundle $\pi^*\mathbb{P}(E)$ is trivial, i.e., it is isomorphic to $\widetilde{X} \times \mathbb{P}(V)$, equipped with the trivial connection, where the vector space V is the typical fiber of the flat vector bundle π^*E. The line subbundle π^*L of π^*E defines a section of the projective bundle $\widetilde{X} \times \mathbb{P}(V)$; the line bundle L is defined in (2.4). Let

$$\gamma : \widetilde{X} \to \mathbb{P}(V) \tag{2.20}$$

be the map defining the section given by π^*L.

The Galois group Γ has a natural action on $\mathbb{P}(V)$ constructed from the combination of the above mentioned trivialization of $\pi^*\mathbb{P}(E)$ using the connection and the fact that $\pi^*\mathbb{P}(E)$ is the pull back of a projective bundle over $\widetilde{X}/\Gamma$. The map γ is evidently equivariant for the actions of Γ on $\widetilde{X}$ and $\mathbb{P}(V)$.

The Assumption 2.4 is equivalent to the condition that γ is *locally nondegenerate* everywhere in the sense that for any $y \in \widetilde{X}$, the order of contact of $\widetilde{X}$ at $\gamma(y)$ with any hyperplane in $\mathbb{P}(V)$ passing through y is at most $r-2$. In particular, the map γ is an immersion. Note that given any Riemann surface C embedded in a projective space $\mathbb{CP}^{r-1}$, and any $p \in C$, there is a hyperplane H in $\mathbb{CP}^{r-1}$ passing through p such that the order of contact of H with C at p is at least $r-2$. Alternative description of locally nondegenerate maps will be given in Section 5.

Another locally nondegenerate map $\tau : \widetilde{X} \to \mathbb{P}(W)$ will be called *equivalent* to γ if there is a linear isomorphism $f : \mathbb{P}(V) \to \mathbb{P}(W)$ such that $f \circ \gamma = \tau$.

Conversely, suppose that we are given a pair (γ, ρ), where ρ is a homomorphism of Γ into $\mathrm{PGL}(V)$ and γ is a holomorphic map of $\widetilde{X}$ into $\mathbb{P}(V)$ which intertwines the actions of Γ. Such a pair will be called a Γ-equivariant map. The trivial projective bundle $\widetilde{X} \times \mathbb{P}(V)$ over $\widetilde{X}$ descends as a flat projective bundle over X.

Further assume that γ is nondegenerate everywhere in the above sense. Then the flat vector bundle over X constructed above from (γ, ρ) satisfies Assumption 2.4. The map of $\widetilde{X}$ to a projective space, constructed in (2.20), corresponding to this flat connection is equivalent to the map γ we started with.

Two flat connections (E, ∇) and (E', ∇') satisfying Assumption 2.4 will be called *equivalent* if there is an isomorphism between the projective bundles $\mathbb{P}(E)$ and $\mathbb{P}(E')$ which preserves the flat connections induced by ∇ and ∇' respectively.

The above construction gives a bijective correspondence between the space of all isomorphism classes of flat connections satisfying Assumption 2.4 and the space of all equivalence classes of Γ-equivariant locally nondegenerate maps of $\widetilde{X}$ in $\mathbb{P}(V)$.

We will return to the locally nondegenerate maps in the final section. Some interesting results on curves embedded in projective spaces, or more generally in Grassmannians, can be found in [8].

3. Construction of flat connections with a special second fundamental form

Before we are able to construct explicit examples of flat connections satisfying Assumption 2.4, we need to recall the notion of a projective structure on a Riemann surface.

A *projective atlas* on a Riemann surface X is a covering $\{U_\alpha, \phi_\alpha\}_{\alpha \in I}$ of X by holomorphic coordinate charts, where ϕ_α is a biholomorphism from the open set U_α of X to an open set in $\mathbb{C}$, such that the map $\phi_\beta \circ \phi_\alpha^{-1}$, for any pair $\alpha, \beta \in I$, is the restriction of a Möbius transformation to the image $\phi_\alpha(U_\alpha \cap U_\beta)$. A Möbius transformation is a function of the form $z \mapsto (az+b)/(cz+d)$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. Another projective atlas $\{U_\alpha, \phi_\alpha\}_{\alpha \in I'}$ is called *equivalent* to $\{U_\alpha, \phi_\alpha\}_{\alpha \in I}$ if their union $\{U_\alpha, \phi_\alpha\}_{\alpha \in I \cup I'}$ is again a projective atlas. A *projective structure* on X is an equivalence class of projective atlases.

The data $\{U_\alpha \cap U_\beta\,,\ \phi_\beta \circ \phi_\alpha^{-1}\}_{\alpha,\beta\in I}$, above, evidently forms a one cocycle with values in $\mathrm{PSL}(2,\mathbb{C})$.

Let $\mathfrak{P}(X)$ denote the space of all projective structures on X. The space $\mathfrak{P}(X)$ is an affine space for the vector space $H^0(X,\ K_X^{\otimes 2})$ [5,4].

Fix a line bundle ξ over the compact Riemann surface X such that $\xi^{\otimes 2}$ is isomorphic to T_X. Fix an isomorphism between $\xi^{\otimes 2}$ and T_X. Using this ξ, for any projective structure on X, the above mentioned cocycle with values in $\mathrm{PSL}(2,\mathbb{C})$ has a natural lift to a cocycle with values in $\mathrm{SL}(2,\mathbb{C})$ [5].

Let the compact Riemann surface X be equipped with a projective structure $\mathcal{P}$. For any $n \geqslant 0$, in [2, Theorem 3.7] a flat connection on the vector bundle $J^n(\xi^{\otimes n})$ was constructed. We will now briefly recall the construction of this connection. It will be shown that the Assumption 2.4 is valid for this connection.

Let V_0 be a two dimensional complex vector space; the line bundle over the projective line $\mathbb{P}(V_0)$ with the tautological quotients of V_0 as the fibers, will be denoted by $\mathcal{O}(1)$. For $n \geqslant 0$, the vector space $H^0(\mathbb{P}(V_0),\ \mathcal{O}(n))$ is naturally identified with the n-fold symmetric tensor product $S^n(V_0)$. Furthermore, the restriction of global sections of $\mathcal{O}(n)$ to n-th order infinitesimal neighborhood of points induces an isomorphism of $J^n(\mathcal{O}(n))$ with the trivial vector bundle over $\mathbb{P}(V_0)$ with $H^0(\mathbb{P}(V_0),\ \mathcal{O}(n))$ as the fiber. In particular, $J^n(\mathcal{O}(n))$ is equipped with a flat connection. This flat connection satisfies Assumption 2.4. Indeed, denoting by $\mathcal{V}$ the trivial rank two vector bundle over $\mathbb{P}(V_0)$ with V_0 as the fiber, consider its line subbundle L_0 that fits in the universal exact sequence

$$0 \longrightarrow L_0 \xrightarrow{\rho} \mathcal{V} \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

Take L in Assumption 2.4 to be the line subbundle $L_0^{\otimes n}$ of $J^n(\mathcal{O}(n))$, given by the image of $\rho^{\otimes n}$ in $S^n(\mathcal{V})$, after invoking the above identification of $J^n(\mathcal{O}(n))$ with $S^n(\mathcal{V})$. It is straight-forward to check that $L_0^{\otimes n}$ has the property required in Assumption 2.4. We note that the line subbundle $L_0^{\otimes n}$ of $J^n(\mathcal{O}(n))$ is the line subbundle $K_{\mathbb{P}(V_0)}^{\otimes n} \otimes \mathcal{O}(n)$ of $J^n(\mathcal{O}(n))$ in the exact sequence (2.7).

The natural action of $\mathrm{SL}(V_0)$ on $\mathbb{P}(V_0)$ lifts to an action on the line bundle $\mathcal{O}(n)$, and hence it lifts to an action on $J^n(\mathcal{O}(n))$. This action of $\mathrm{SL}(V_0)$ clearly preserves the natural flat connection on $J^n(\mathcal{O}(n))$.

Suppose that we have a projective structure $\mathcal{P}$ on X defined by a coordinate atlas $\{U_\alpha, \phi_\alpha\}_{\alpha\in I}$ such that the target of each map ϕ_α is $\mathbb{P}(V_0)$. Further assume that for any pair $\alpha, \beta \in I$, we are given an element $G_{\alpha,\beta} \in \mathrm{SL}(V_0)$ such that the transition function $\phi_\alpha \circ \phi_\beta^{-1}$ is the restriction to $\phi_\beta(U_\beta) \subset \mathbb{P}(V_0)$ of the action of $G_{\alpha,\beta}$ on $\mathbb{P}(V_0)$, and also assume that $\{G_{\alpha,\beta}\}_{\alpha,\beta\in I}$ is a one cocycle. As it was noted earlier, the choice of ξ enables us to take the cocycle defining the projective structure to be $\mathrm{SL}(V_0)$ valued.

Consider the line bundle $\phi_\alpha^*\mathcal{O}(n)$ over U_α, where $\alpha \in I$. For any pair $\alpha, \beta \in I$, the action of $G_{\alpha,\beta}$ on $\mathcal{O}(n)$, for the natural action of $\mathrm{SL}(V_0)$ on $\mathcal{O}(n)$, can be considered as the clutching function. The resulting vector bundle over X is $\xi^{\otimes n}$.

Since the action of $\mathrm{SL}(V_0)$ on $J^n(\mathcal{O}(n))$ preserves the natural flat connection on it, using the coordinate charts compatible with the projective structure $\mathcal{P}$, the connection on $J^n(\mathcal{O}(n))$ induces a flat connection on $J^n(\xi^{\otimes n})$; the connection on $J^n(\xi^{\otimes n})$ will be denoted by $\nabla^{\mathcal{P}}$. Consider the line subbundle

$$K_X^{\otimes n} \otimes \xi^{\otimes n} \subset J^n\big(\xi^{\otimes n}\big),$$

defined in (2.7), which will be denoted by L_n. Since the line subbundle $L_0^{\otimes n}$ of the flat vector bundle $J^n(\mathcal{O}(n))$ over $\mathbb{P}(V_0)$ satisfies Assumption 2.4, the line subbundle L_n of $J^n(\xi^{\otimes n})$ over X satisfies Assumption 2.4 for the flat connection $\nabla^{\mathcal{P}}$. Indeed, with respect to a coordinate chart on X compatible with the projective structure $\mathcal{P}$, the flat connection on $J^n(\xi^{\otimes n})$ is exactly the canonical flat connection on $J^n(\mathcal{O}(n))$.

Since $J^n(\xi^{\otimes n})$ admits a holomorphic connection satisfying Assumption 2.4, from Lemma 2.16 it follows that any holomorphic connection on $J^n(\xi^{\otimes n})$ satisfies Assumption 2.4. The Proposition 2.6 describes the extension classes for the filtration of $J^n(\xi^{\otimes n})$ obtained using (2.7).

We will show at the end of this section how to recover the projective structure $\mathcal{P}$ from the flat connection $\nabla^{\mathcal{P}}$ on $J^n(\xi^{\otimes n})$ constructed above from $\mathcal{P}$.

For any two vector bundle V and V' over X, the sheaf of sections of the vector bundle

$$\mathrm{Diff}^n_X(V, V') := \mathrm{Hom}\big(J^n(V), V'\big)$$

is called the sheaf of *differential operators of order n from V to V'*.

The restriction homomorphism

$$\sigma : \mathrm{Diff}^n_X(V, V') \to \mathrm{Hom}\big(K_X^{\otimes n} \otimes V, V'\big) \tag{3.1}$$

obtained from (2.7) is known as the *symbol map*.

Let D be a holomorphic connection on $J^n(\xi^{\otimes n})$.

Setting $i = n + 1$ in (2.8), an injective homomorphism

$$\Phi : J^n\big(\xi^{\otimes n}\big) \to J^{n+1}\big(\xi^{\otimes n}\big) \tag{3.2}$$

is obtained. The following diagram

$$\begin{array}{ccc} E & \xrightarrow{\Phi(i+1)} & J^{i+1}(\mathcal{L}) \\ \| & & \downarrow \\ E & \xrightarrow{\Phi(i)} & J^{i}(\mathcal{L}) \end{array}$$

is commutative, where $\Phi(i)$ is as in (2.8). For the given connection D, the homomorphism $\Phi(n)$ is an isomorphism (Theorem 2.9). This implies that the image of the homomorphism Φ, defined in (3.2), is transversal to the image of $K_X^{\otimes(n+1)} \otimes \xi^{\otimes n}$ in $J^{n+1}(\xi^{\otimes n})$ by the homomorphism defined in (2.7).

Thus using the image of Φ we have a splitting of the canonical exact sequence of jets:

$$0 \to \xi^{\otimes(-2-n)} = K_X^{\otimes(n+1)} \otimes \xi^{\otimes n} \to J^{n+1}\big(\xi^{\otimes n}\big) \to J^n\big(\xi^{\otimes n}\big) \to 0.$$

We note that the homomorphism Φ itself need not be a splitting map. In other words, the composition of Φ with the projection

$$J^{n+1}\big(\xi^{\otimes n}\big) \to J^n\big(\xi^{\otimes n}\big)$$

need not be the identity map of $J^n(\xi^{\otimes n})$.

This splitting of the jet sequence gives a homomorphism:

$$\mathcal{D}(n+1) : J^{n+1}\big(\xi^{\otimes n}\big) \to \xi^{\otimes(-2-n)} = K_X^{\otimes(n+1)} \otimes \xi^{\otimes n}, \tag{3.3}$$

such that the composition of $\mathcal{D}(n+1)$ with the inclusion of $K_X^{\otimes(n+1)} \otimes \xi^{\otimes n}$ in $J^{n+1}(\xi^{\otimes n})$ is the identity map of $K_X^{\otimes(n+1)} \otimes \xi^{\otimes n}$.

Therefore, the homomorphism $\mathcal{D}(n+1)$ constructed in (3.3) is a global differential operator

$$\mathcal{D} \in H^0\big(X, \mathrm{Diff}_X^{n+1}\big(\xi^{\otimes n}, \xi^{\otimes(-2-n)}\big)\big)$$

whose symbol is the constant function 1; note that the symbol of $\mathcal{D}$ is a section of $T_X^{n+1} \otimes \xi^{\otimes -n} \otimes \xi^{\otimes(-2-n)} = \mathcal{O}_X$.

Now we introduce the two main objects of study here:

Let $\mathcal{A}$ denote the space of all flat connections on $J^n(\xi^{\otimes n})$.

Let $\mathcal{B}$ denote the space of all global differential operators

$$\delta \in H^0\big(X,\ \mathrm{Diff}_X^{n+1}\big(\xi^{\otimes n}, \xi^{\otimes(-2-n)}\big)\big)$$

such that the symbol $\sigma(\delta) \in H^0(X, \mathrm{Hom}(\xi^{\otimes n} \otimes T_X^{\otimes(n+1)}, \xi^{\otimes(-2-n)})) = H^0(X, \mathcal{O}_X)$ is the constant function 1.

Let

$$\mathcal{F} : \mathcal{A} \to \mathcal{B} \tag{3.4}$$

be the map that sends any connection D to the operator $\mathcal{D}$ constructed from D in the above fashion.

Since $\xi^{\otimes 2} = T_X$, we have:

$$\bigwedge^{n+1} J^n\big(\xi^{\otimes n}\big) = \mathcal{O}_X.$$

Let $\mathcal{A}_0 \subset \mathcal{A}$ be the subset consisting of all flat connections such that the induced connection on $\bigwedge^{n+1} J^n(\xi^{\otimes n})$ is the trivial connection on $\mathcal{O}_X$. This is equivalent to the condition that the induced connection on $\bigwedge^{n+1} J^n(\xi^{\otimes n})$ has trivial monodromy. The connection $\nabla^{\mathcal{P}}$ constructed earlier from a projective structure $\mathcal{P}$ on X, belongs to $\mathcal{A}_0$. Indeed, this is an immediate consequence of the fact that the natural connection on the vector bundle $J^n(\mathcal{O}(n))$ over the projective line induces the trivial connection on $\bigwedge^{n+1} J^n(\mathcal{O}(n)) = \mathcal{O}$.

The image $\mathcal{F}(\mathcal{A}_0) \subset \mathcal{B}$ will be denoted by $\mathcal{B}_0$. The differential operators in the class $\mathcal{B}_0$ have the following description. For any holomorphic coordinate function (U, z) on X, let $(\frac{\partial}{\partial z})^{1/2}$ denote a section of local of ξ such that the local section $(\frac{\partial}{\partial z})^{1/2} \otimes (\frac{\partial}{\partial z})^{1/2}$ of $\xi^{\otimes 2} = T_X$ is the local section $\frac{\partial}{\partial z}$ of T_X. The corresponding local section of $\xi^{\otimes j}$, where $j \in \mathbb{Z}$, will be denoted by $(\frac{\partial}{\partial z})^{j/2}$. The description of any operator $\delta \in \mathcal{B}_0$ in terms of local coordinates on X is:

$$\begin{aligned}\delta\left(f\left(\frac{\partial}{\partial z}\right)^{n/2}\right) = \bigg(&\frac{\mathrm{d}^{n+1} f}{\mathrm{d}z^{n+1}} + a_2 \frac{\mathrm{d}^{n-1} f}{\mathrm{d}z^{n-1}} + a_3 \frac{\mathrm{d}^{n-2} f}{\mathrm{d}z^{n-2}} + \cdots \\ &+ a_i \frac{\mathrm{d}^{n-i+1} f}{\mathrm{d}z^{n-i+1}} + \cdots + a_{n+1}\bigg)\left(\frac{\partial}{\partial z}\right)^{(-2-n)/2},\end{aligned} \tag{3.5}$$

where f and a_i are local holomorphic functions. The point to note is that $a_1 = 0$.

Choose a projective structure on X. Setting $k = -n$ and $l = n+2$ in [2, Section 1, Corollary C] we have the decomposition

$$H^0\big(X,\ \mathrm{Diff}_X^{n+1}\big(\xi^{\otimes n}, \xi^{\otimes(-2-n)}\big)\big) = \bigoplus_{i=0}^{n+1} H^0\big(X,\ K_X^{\otimes i}\big) \tag{3.6}$$

of the space of differential operators. This decomposition is constructed using the given projective structure. The subset $\mathcal{B}_0$ of $H^0(X,\ \mathrm{Diff}_X^{n+1}(\xi^{\otimes n}, \xi^{\otimes(-2-n)}))$ corresponds to the subset of the right hand side of (3.6) consisting of all elements of the form $1+\sum_{j=2}^{n+1}\omega_j$, where $\omega_j \in H^0(X, K_X^{\otimes j})$.

Given any $\delta \in \mathcal{B}_0$, there is a unique projective structure $\mathcal{P}'$ on X such that for the decomposition (3.6) constructed using the projective structure $\mathcal{P}'$, the component of δ in $H^0(X, K_X^{\otimes 2})$ in the decomposition vanishes. The coordinate charts on X for the projective structure $\mathcal{P}'$ are determined by the condition that the local function a_2 in (3.5) vanishes identically in the local description of δ.

Therefore, we have a natural map

$$\Psi : \mathcal{B}_0 \to \mathfrak{P}(X), \tag{3.7}$$

where $\mathfrak{P}(X)$ denotes, as before, the space of all projective structures on X, which sends any differential operator δ to the projective structure $\mathcal{P}'$ constructed above.

For a projective structure $\mathcal{P} \in \mathfrak{P}(X)$, consider the flat connection $\nabla^{\mathcal{P}}$ on $J^n(\xi^{\otimes n})$ constructed earlier. It is easy to check that

$$\Psi \circ \mathcal{F}\big(\nabla^{\mathcal{P}}\big) = \mathcal{P}. \tag{3.8}$$

Therefore, the flat connection $\nabla^{\mathcal{P}}$ determines the projective structure $\mathcal{P}$.

4. The moduli space of all flat connections with the special second fundamental form

The aim in this section is to obtain a description of the space of all isomorphism classes of flat connections on $J^n(\xi^{\otimes n})$. Note that this space is same as the quotient space for the natural action of the group of holomorphic automorphisms of $J^n(\xi^{\otimes n})$ on $\mathcal{A}$.

For any integer $j \geqslant 0$, the projection $J^{i+j}(V) \to J^i(V)$ obtained from (2.7) has a natural lift $\tau : J^{i+j}(V) \to J^j(J^i(V))$ such that the following diagram of homomorphisms commute

$$\begin{array}{ccc} J^{i+j}(V) & \xrightarrow{\tau} & J^j(J^i(V)) \\ \downarrow & & \downarrow \\ J^i(V) & = & J^i(V) \end{array} . \tag{4.1}$$

The above projection $J^j(J^i(V)) \to J^i(V)$ is the evaluation map, i.e., the one obtained by setting $n=0$ in (2.7).

Now consider the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \xi^{\otimes(-2-n)} & \longrightarrow & J^{n+1}(\xi^{\otimes n}) & \longrightarrow & J^n(\xi^{\otimes n}) & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow \tau & & \| & & \\ 0 & \longrightarrow & K_X \otimes J^n(\xi^{\otimes n}) & \longrightarrow & J^1(J^n(\xi^{\otimes n})) & \longrightarrow & J^n(\xi^{\otimes n}) & \longrightarrow & 0 \end{array} . \tag{4.2}$$

The homomorphism τ is defined in (4.1); the left vertical map f is the inclusion obtained from (2.7). Given any splitting of the top exact sequence in (4.2), composing τ with the homomorphism $J^n(\xi^{\otimes n}) \to J^{n+1}(\xi^{\otimes n})$, corresponding to the given splitting, we get a splitting of the bottom exact sequence in (4.2).

A splitting of the bottom exact sequence in (4.2) is a holomorphic connection on $J^n(\xi^{\otimes n})$ [1]. Thus, using (4.2), we obtain a map:

$$\mathcal{F}' : \mathcal{B} \to \mathcal{A}. \tag{4.3}$$

Recall that the spaces $\mathcal{A}$ and $\mathcal{B}$ were defined in Section 3.

For any differential operator $\delta \in \mathcal{B}$, and any vector $v \in J^n(\xi^{\otimes n})_x$, there is a unique local section s of $\xi^{\otimes n}$, defined in a neighborhood of x, such that $\delta(s) = 0$ and $s(x) = v$. Using this fact it is easy to construct a flat connection on $J^n(\xi^{\otimes n})$ which is characterized by the fact that its flat local sections are precisely the ones obtained from the sections s of $\xi^{\otimes n}$ of the above type. This flat connection on $J^n(\xi^{\otimes n})$ coincides with the flat connection $\mathcal{F}'(\delta)$. The homomorphism defining the splitting of the top exact sequence in (4.2) for δ sends the vector v to the vector in $J^{n+1}(\xi^{\otimes n})_x$ representing s.

In (3.4) a map, $\mathcal{F}$, from $\mathcal{A}$ to $\mathcal{B}$ was constructed. The following simple proposition is deduced from the construction of the maps $\mathcal{F}$ and $\mathcal{F}'$.

PROPOSITION 4.4. – *The composition $\mathcal{F} \circ \mathcal{F}'$ is the identity map of $\mathcal{B}$.*

Proof. – Take any differential operator $\delta \in \mathcal{B}$. For $x \in X$ and $v \in J^n(\xi^{\otimes n})_x$, let s be the unique local section of $\xi^{\otimes n}$ defined in a neighborhood of x such that $\delta(s) = 0$ and the element of $J^n(\xi^{\otimes n})_x$ representing s is v. It was noted earlier that the image of s in $J^{n+1}(\xi^{\otimes n})_x$ coincides with the image of v by the splitting of the top exact sequence in (4.2) defining δ.

Given a flat connection on $J^n(\xi^{\otimes n})$, the image of v in $J^1(J^n(\xi^{\otimes n}))_x$, by the splitting of the bottom exact sequence in (4.2) corresponding to the flat connection, represents the unique flat section s' of $J^n(\xi^{\otimes n})$ defined in a neighborhood of x which takes the value v at x.

Now, set $\mathcal{F}'(\delta)$ to be the flat connection in the above remark. Consider the flat section s', of $J^n(\xi^{\otimes n})$, constructed above using $\mathcal{F}'(\delta)$ for any $v \in J^1(J^n(\xi^{\otimes n}))_x$. This section s' represents the section s of $\xi^{\otimes n}$, constructed above, for the differential operator δ. From these observations the proposition follows. □

The above proposition implies that $\mathcal{F}$ is surjective and $\mathcal{F}'$ is injective; the injectivity of $\mathcal{F}'$, of course, is an immediate consequence of the injectivity of the homomorphism τ in (4.2). The map $\mathcal{F}'$ cannot be surjective from dimension considerations.

Let $\mathcal{G}$ denote the group $\mathrm{Aut}(J^n(\xi^{\otimes n}))$ consisting of all holomorphic automorphisms of $J^n(\xi^{\otimes n})$. The space $\mathcal{A}$ admits a natural left action of $\mathcal{G}$. More precisely, the action of any $g \in \mathcal{G}$ sends a connection operator D to the connection operator $g \circ D \circ g^{-1}$. The group $\mathcal{G}$ is large in the sense that if we fix a projective structure on X, then there is a natural inclusion of the Cartesian product

$$\mathbb{C}^* \times \left(\bigoplus_{i=1}^{n} H^0\big(X,\, K_X^{\otimes i}\big) \right)$$

into $\mathcal{G}$. This remark will be explained later.

LEMMA 4.5. – *For any $\delta \in \mathcal{B}$, the preimage $\mathcal{F}^{-1}(\delta) \subset \mathcal{A}$ is left invariant by the action of $\mathcal{G}$ on $\mathcal{A}$.*

Proof. – From Corollary 2.17 we know that the vector bundle $J^n(\xi^{\otimes n})$ is indecomposable. Consequently, for any $g \in \mathcal{G}$ and $x \in X$, all the eigenvalues of the endomorphism $g(x)$ of $J^n(\xi^{\otimes n})_x$ coincide. Furthermore, this common eigenvalue of $g(x)$ is independent of x. Let the constant common eigenvalue of g be denoted by λ_g.

Let $\phi : J^n(\xi^{\otimes n}) \to \xi^{\otimes n}$ be the evaluation map, as in (2.7) with $n = 0$.

The automorphism g preserves the Harder–Narasimhan filtration of $J^n(\xi^{\otimes n})$. Recall that the Harder–Narasimhan filtration of $J^n(\xi^{\otimes n})$ is obtained from (2.14). Hence, in particular, for any local section s to $J^n(\xi^{\otimes n})$, the following identity

$$\phi \circ g(s) = \lambda_g \cdot \phi(s) \tag{4.6}$$

is valid.

Take any $\delta \in \mathcal{B}$ and any $D \in \mathcal{F}^{-1}(\delta)$. According to the construction of the map $\mathcal{F}$, the differential operator $\mathcal{F}(D)$ is the one given by the splitting of the top exact sequence in (4.2) determined by the following condition: at any $x \in X$, the image of the homomorphism

$$S_D(x) : J^n\left(\xi^{\otimes n}\right)_x \to J^{n+1}\left(\xi^{\otimes n}\right)_x,$$

defining the splitting, consists of all vectors representing sections of $\xi^{\otimes n}$ of the form $\phi(u)$, where u is a flat section of $J^n(\xi^{\otimes n})$ for the connection D.

The local section s is a flat section for the connection D if and only if $g(s)$ is a flat section for the connection $g \circ D \circ g^{-1}$. Consequently, using the description of the differential operator $\mathcal{F}(D)$ given above, it follows from the identity (4.6) that the image of the splitting homomorphism S_D for $\mathcal{F}(D)$ coincides with the image of the splitting homomorphism $S_{g\circ D\circ g^{-1}}$ for the differential operator $\mathcal{F}(g \circ D \circ g^{-1})$. This is equivalent to the assertion that $\mathcal{F}(g \circ D \circ g^{-1}) = \mathcal{F}(D) = \delta$. This completes the proof of the lemma. $\square$

As the map $\mathcal{F}$ is surjective, the above lemma implies that there is a unique surjective map

$$P : \mathcal{A}/\mathcal{G} \to \mathcal{B} \tag{4.7}$$

such that $\mathcal{F} = P \circ Q$, where Q denotes the quotient map for the action of $\mathcal{G}$.

The remark prior to Lemma 4.5 that $\mathbb{C}^* \times (\bigoplus_{i=1}^n H^0(X,\, K_X^{\otimes i}))$ sits naturally inside $\mathcal{G}$ using a projective structure on X, will be quickly explained.

Once a projective structure $\mathcal{P}$ on X fixed, the vector bundle $J^n(\xi^{\otimes n})$ gets identified with the symmetric tensor product $S^n(J^1(\xi))$ [2, Section 1, Theorem A]. On the projective line $\mathbb{CP}^1$, there is a natural identification of $J^n(\mathcal{O}(n))$ with $S^n(J^1(\mathcal{O}(1)))$. This isomorphism induces the above isomorphism on X using the projective structure. For $x \in X$, consider the exact sequence

$$0 \longrightarrow (K_X \otimes \xi)_x \xrightarrow{a_x} J^1(\xi)_x \xrightarrow{b_x} \xi_x \longrightarrow 0$$

as in (2.7). This gives an inclusion $m_x : K_x \to \operatorname{End}(J^1(\xi)_x)$, which sends any $v \in K_x$ to the endomorphism defined by $w \mapsto a_x(v \otimes b_x(w))$. Therefore, m_x gives an injective homomorphism of K_x into $\operatorname{End}(S^n(J^1(\xi)_x))$. Using the above mentioned identification of $J^n(\xi^{\otimes n})$ with $S^n(J^1(\xi))$, the action m_x of K_x on $J^n(\xi^{\otimes n})_x$ actually sends it to the subspace $(F_n)_x$, the fiber of the previous term of the Harder–Narasimhan filtration as in (2.5). More generally, the subspace $(F_j)_x$ is mapped onto $(F_{j-1})_x$. Therefore, the $(n+1)$-times iteration $(m_x)^{n+1}$ is the zero homomorphism of $K_x^{\otimes(n+1)}$ into $\operatorname{End}(S^n(J^1(\xi)_x))$. The homomorphism

$$\bigoplus_{j=1}^{n}(m_x)^j : \bigoplus_{j=1}^{n} K_x^{\otimes j} \to \operatorname{End}\left(S^n\left(J^1(\xi)_x\right)\right)$$

is evidently injective. Thus, we have an inclusion of $\bigoplus_{i=1}^n H^0(X,\, K_X^{\otimes i})$ into $\mathcal{G}$ whose restriction to any $x \in X$ coincides with the above homomorphism. It is easy to see that the image is contained in the nilpotent part of $\mathcal{G}$. The scalars $\mathbb{C}^*$ acts by multiplication on $S^n(J^1(\xi))$. Thus we obtain the injective map

$$\Sigma_{\mathcal{P}} : \mathbb{C}^* \times \left(\bigoplus_{i=1}^{n} H^0\big(X,\ K_X^{\otimes i}\big) \right) \to \mathcal{G} \tag{4.8}$$

that we are seeking. Since the identification of $J^n(\xi^{\otimes n})$ with $S^n(J^1(\xi))$ depends on the choice of the projective structure $\mathcal{P}$, the above map $\Sigma_{\mathcal{P}}$ depends on $\mathcal{P}$. It can be checked that the map $\Sigma_{\mathcal{P}}$ is bijective, but we will not need this fact here.

In Proposition 2.18 we saw that any connection $D \in \mathcal{A}$ is irreducible. Therefore, using the Schur's lemma it follows immediately that the subgroup of $\mathcal{G}$ that fixes the point D actually coincides with the group of nonzero scalar multiplications. Now from the existence of the injective map $\Sigma_{\mathcal{P}}$ in (4.8) it follows that the dimension of any orbit for the action of $\mathcal{G}$ on $\mathcal{A}$ is at least $\sum_{i=1}^{n} \dim H^0(X,\ K_X^{\otimes i}) = 1 + n^2(g-1)$.

THEOREM 4.9. – *The map P constructed in* (4.7) *is bijective. In other words, the quotient space $\mathcal{A}/\mathcal{G}$ gets identified with $\mathcal{B}$.*

Proof. – Since the map P in (4.7) has already been proved to be surjective, it is enough to show that P is injective. In view of Lemma 4.5, it suffices to show that the group $\mathcal{G}$ acts transitively on $\mathcal{F}^{-1}(\delta)$ for every $\delta \in \mathcal{B}$.

Take a flat connection $D \in \mathcal{A}$ on $J^n(\xi^{\otimes n})$. Let D_0 denote the flat connection $\mathcal{F}' \circ \mathcal{F}(D) \in \mathcal{A}$, where $\mathcal{F}'$ and $\mathcal{F}$ are defined in (4.3) and (3.4) respectively.

Let

$$G : J^n\big(\xi^{\otimes n}\big) \to J^n\big(\xi^{\otimes n}\big)$$

be the isomorphism obtained by setting $(J^n(\xi^{\otimes n}), D)$ to be the flat vector bundle in Theorem 2.9.

From the construction of the automorphism G it is easy to deduce that the action of G on $\mathcal{A}$ sends the connection D to the connection D_0. In particular, D and D_0 are in the same $\mathcal{G}$ orbit in $\mathcal{A}$.

Now, from Proposition 4.4 it follows immediately that $\mathcal{G}$ acts transitively on the inverse image $\mathcal{F}^{-1}(\mathcal{F}(D))$. This completes the proof of the theorem. □

We note that the image $\mathcal{F}'(\mathcal{B})$ can be characterized in the following way. The subset $\mathcal{F}'(\mathcal{B})$ of $\mathcal{A}$ consists of all flat connections D such that the automorphism G constructed for D in the proof of Theorem 4.9 coincides with the identity automorphism of $J^n(\xi^{\otimes n})$.

Theorem 4.9 and Proposition 4.4 combine together to give the following corollary:

COROLLARY 4.10. – *Let*

$$P' := Q \circ \mathcal{F}' : \mathcal{B} \to \mathcal{A}/\mathcal{G}$$

be the composition map, where $\mathcal{F}'$ is defined in (4.3)*, and $Q : \mathcal{A} \to \mathcal{A}/\mathcal{G}$ as before is the quotient map. The map P' is bijective, and it coincides with the inverse of the map P.*

Recall that $\mathcal{A}_0$ is the subset of $\mathcal{A}$ consisting of all connections with the property that the canonical isomorphism of $\bigwedge^{n+1} J^n(\xi^{\otimes n})$ with $\mathcal{O}_X$ takes the induced connection on $\bigwedge^{n+1} J^n(\xi^{\otimes n})$ to the trivial connection on $\mathcal{O}_X$. The action of $\mathcal{G}$ on $\mathcal{A}$ evidently leaves invariant the subset $\mathcal{A}_0$. From Theorem 4.9 it follows that the restriction of P to $\mathcal{A}_0/\mathcal{G}$ is a bijection to $\mathcal{B}_0 := \mathcal{F}(\mathcal{A}_0)$. Similarly, from Corollary 4.10, it follows that the restriction of the map P' to $\mathcal{B}_0$ is also bijective.

Since any connection $D \in \mathcal{A}$ is irreducible (Proposition 2.18), there is a stable Higgs bundle over X of degree zero associated to D [9]. It would be very interesting to be able to determine the class of stable Higgs bundles that arise this way.

If we set $n = 1$, then the map Ψ from $\mathcal{B}_0$ to $\mathfrak{P}(X)$ constructed in (3.7) is bijective. Let $\mathcal{P}$ denote the projective structure on X given by the uniformization theorem. Hitchin had shown in [6] that the Higgs bundles corresponding to the flat connection $\mathcal{F}' \circ \Psi^{-1}(\mathcal{P})$ is $(\xi \oplus \xi^*, \theta)$, where the Higgs field θ is of the form

$$\theta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Using the results obtained in this section we will construct invariants in the next section for the Γ-equivariant everywhere locally nondegenerate maps of $\widetilde{X}$ into $\mathbb{CP}^n$ considered in Section 2.

5. Invariants for everywhere locally nondegenerate maps

Let

$$0 \longrightarrow S \longrightarrow V \xrightarrow{q} Q \longrightarrow 0 \tag{5.1}$$

be the universal exact sequence over $\mathbb{CP}^n$, the projective space of lines in $\mathbb{C}^{n+1}$. The vector bundle V is the trivial vector bundle with $\mathbb{C}^{n+1}$ as fiber.

As in Section 2, let $\pi : \widetilde{X} \to X$ be a universal cover with Galois group Γ. Let

$$\gamma : \widetilde{X} \to \mathbb{CP}^n \tag{5.2}$$

be a holomorphic map. Consider the differential of the map γ:

$$\mathrm{d}\gamma : T_{\widetilde{X}} \to \gamma^* T_{\mathbb{CP}^n} = \gamma^* \mathrm{Hom}(S, Q).$$

Assume that γ is an immersion, i.e., $\mathrm{d}\gamma$ is an injective homomorphism of vector bundles.

The homomorphism $\mathrm{d}\gamma$ gives a homomorphism

$$\overline{\mathrm{d}\gamma} : T^*_{\widetilde{X}} \otimes \gamma^* S \to \gamma^* Q. \tag{5.3}$$

Let S_1 denote the inverse image $q^{-1}(\mathrm{image}(\overline{d\gamma}))$, where q is defined in (5.1). The subbundle S_1 of $\gamma^* V$ defines a map

$$\gamma_1 : \widetilde{X} \to G(n+1, 2)$$

of $\widetilde{X}$ into the Grassmannian of two planes.

Now assume that γ_1 is an immersion. Then repeating the above argument we get a map

$$\gamma_2 : \widetilde{X} \to G(n+1, 3).$$

of $\widetilde{X}$ into the Grassmannian of three planes.

More generally, inductively we have a map

$$\gamma_i : \widetilde{X} \to G(n+1, i+1),$$

where $i \in [1, n-1]$, by assuming that γ_{i-1} is an immersion. See Section 1 of [8] for the details of the construction of the maps γ_i described above.

Let $\rho : \Gamma \to \mathrm{GL}(n+1, \mathbb{C})$ be a homomorphism, and let the map γ in (5.2) be equivariant for the actions of Γ on $\widetilde{X}$ and $\mathrm{GL}(n+1, \mathbb{C})$; the group Γ acts on $\widetilde{X}$ as deck transformations and it acts on $\mathbb{CP}^n$ using the linear representation ρ. Therefore, the natural flat connection on the trivial vector bundle $\gamma^* V$ descends as a flat connection, which will be denoted by D, on the vector bundle E over X associated to ρ.

Two such pairs (γ, ρ) and (γ', ρ') will be called *equivalent* if there is a linear automorphism F of $\mathbb{CP}^n$ such that $F \circ \gamma = \gamma'$ and the homomorphism from Γ to $\mathrm{GL}(n+1, \mathbb{C})$ given by $F\rho F^{-1}$ coincides with the homomorphism given by ρ'.

Let $\mathcal{S}(X)$ denote the space of all equivalence classes of pairs (γ, ρ) such that γ is an immersion and also each γ_i, where $i \in [1, n-1]$, is an immersion.

Consider the homomorphism $\gamma^* S \to \pi^* K_X \otimes \pi^* Q$ obtained by tensoring $\overline{\mathrm{d}\gamma}$, constructed in (5.3), with the identity endomorphism of $K_{\widetilde{X}} = \pi^* K_X$. It is easy to see that this homomorphism coincides with the second fundamental form, defined in (2.2), of $\gamma^* S$ for the natural flat connection on the trivial vector bundle $\gamma^* V$. From this observation it follows that the Assumption 2.4 for the flat connection D is equivalent to the assumption that the map γ and also each map γ_i, where $i \in [1, n-1]$, is an immersion.

The condition that the map γ, and also each map γ_i, where $i \in [1, n-1]$, is an immersion, is equivalent to the condition that the map γ is everywhere locally nondegenerate in the sense defined in Section 2.

Tensoring the flat vector bundle E by a line bundle with connection does not change the connection on the projective bundle $\mathbb{P}(E)$. Therefore, from the above observations we conclude that the space $\mathcal{S}(X)$ is canonically identified with the quotient space $\mathcal{A}_0/\mathcal{G}$.

We recall that, by definition of the subset $\mathcal{B}_0$ of $\mathcal{B}$, the map P constructed in (4.7) sends the subset $\mathcal{A}_0$ surjectively to $\mathcal{B}_0$.

Using the decomposition (3.6), and the map Ψ constructed in (3.7), the space $\mathcal{B}_0$ gets identified with the Cartesian product

$$\mathfrak{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0\big(X,\ K_X^{\otimes i}\big) \right),$$

which will be explained next. Take any differential operator $\delta \in \mathcal{B}_0$. As we mentioned earlier, the direct sum decomposition (3.6) of δ with respect to the projective structure $\Psi(\delta)$ is of the form $1 + \sum_{j=3}^{n+1} \omega_j$, where $\omega_j \in H^0(X,\ K_X^{\otimes j})$. Now consider the map

$$F : \mathcal{B}_0 \to \mathfrak{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0\big(X,\ K_X^{\otimes i}\big) \right), \tag{5.4}$$

which sends any δ to $(\Psi(\delta), \sum_{j=3}^{n+1} \omega_j)$. The map F is obviously surjective. From the combination of the decomposition (3.6) together with the identity (3.8) it follows that the above map F is injective.

Now, using the bijective map F defined in (5.4) together with the identification of $\mathcal{S}(X)$ with $\mathcal{A}_0/\mathcal{G}$ established earlier, we have the following reformulation of the map P constructed in (4.7):

THEOREM 5.5. – *There is a canonical bijective map*

$$\mathcal{I} : \mathcal{S}(X) \to \mathfrak{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0\big(X,\ K_X^{\otimes i}\big) \right)$$

from the space of equivalence classes of Γ-equivariant everywhere locally nondegenerate maps of $\widetilde{X}$ into $\mathbb{CP}^n$.

Using the identification of $\mathcal{B}_0$ with $\mathfrak{P}(X) \times (\bigoplus_{i=3}^{n+1} H^0(X,\, K_X^{\otimes i}))$, the map P', constructed in Corollary 4.10, gives the inverse

$$\mathcal{I}^{-1} : \mathfrak{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0\big(X,\, K_X^{\otimes i}\big) \right) \to \mathcal{S}(X)$$

of the map $\mathcal{I}$. Indeed, this is an immediate consequence of Corollary 4.10.

REFERENCES

[1] M.F. ATIYAH, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* 85 (1957) 181–207.

[2] I. BISWAS, Differential operators on complex manifolds with a flat projective structure, *J. Math. Pures Appl.* 78 (1999) 1–26.

[3] I. BISWAS and A.K. RAINA, Projective structures on a Riemann surface, II, *Internat. Math. Res. Notices* (1999) 685–716.

[4] P. DELIGNE, *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Math., Vol. 163, Springer-Verlag, Berlin, 1970.

[5] R.C. GUNNING, *Lectures on Riemann Surfaces*, Mathematical Notes 2, Princeton University Press, Princeton, NJ, 1966.

[6] N.J. HITCHIN, The self-duality equations on a Riemann surface, *Proc. Lond. Math. Soc.* 55 (1987) 59–126.

[7] S. KOBAYASHI, *Differential Geometry of Complex Vector Bundles*, Publications of the Math. Society of Japan 15, Iwanami Shoten Publishers and Princeton University Press, 1987.

[8] D. PERKINSON, Curves in Grassmannians, *Trans. Amer. Math. Soc.* 347 (1995) 3179–3246.

[9] C.T. SIMPSON, Higgs bundles and local systems, *Inst. Hautes Études Sci. Publ. Math.* 75 (1998) 5–95.